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1992 J. Phys. A: Math. Gen. 25 1575

(http://iopscience.iop.org/0305-4470/25/6/017)

J. Phys. A: Math. Gen. 25 (1992) 1575–1591. Printed in the UK

On the existence and uniqueness of dissipative plasma equilibria in a toroidal domain

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Received 1 August 1991, in final form 18 November 1991

Abstract. A one-fluid, dissipative magnetohydrodynamic model of plasma equilibrium in a torus is considered. The equations include inertial forces, finite resistivity and viscosity, and a particle source which sustains the pressure gradient in the plasma; viscosity is described by the Braginskii operator. Plasma density, resistivity and viscosity coefficients are assumed to be uniform. A boundary-value problem in a general toroidal domain is formulated, no further assumption on the domain being made besides a sufficient regularity of its boundary. The system of equations is reduced to a problem with unknowns p, $\boldsymbol{v}$, $\boldsymbol{B}$ (p denotes the scalar pressure, $\boldsymbol{v}$ the flow velocity, $\boldsymbol{B}$ the magnetic field). A functional setting of the equations is established and, generalizing the classical mathematical techniques adopted in the theory of viscous incompressible flow to investigate the solvability of the steady-state Navier–Stokes equations, a problem for weak solutions is formulated which is shown to be equivalent to solving a nonlinear equation in a separable Hilbert space. Then, by analysing the Braginskii viscosity in the established functional framework, we find properties which allow us to write the above equation as a fixed-point equation. The main results of our analysis are the following: (i) we prove the existence of at least one weak solution if the source is sufficiently small, or viscosity and resistivity sufficiently large; (ii) we obtain an estimate of the solution(s); (iii) we prove that, under a condition of the same kind as that for existence, but more stringent, there exists only one solution; and (iv) the well known existence and uniqueness results for the steady-state Navier–Stokes problem are recovered when the magnetic field is set equal to zero.

1. Introduction

The one-fluid, ideal, magnetohydrodynamic (MHD) model is commonly adopted to describe the equilibrium of a plasma contained in a torus. According to the ideal MHD model, the pressure gradient is simply balanced by the magnetic force, i.e. the equations

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B} \qquad \boldsymbol{j} = \nabla \times \boldsymbol{B}$$

hold, where p denotes the scalar pressure, $\boldsymbol{j}$ the current density and $\boldsymbol{B}$ the magnetic field (we assume that $\mu_0 = 1$ throughout this paper). The effect of the plasma flow velocity on the force balance is not taken into account. As is well known, in the presence of axial symmetry solving these equations reduces to solving a two-dimensional elliptic equation, the Lüst–Schlüter–Grad–Shafranov equation.

0305-4470/92/061575+17$04.50

This model has been thoroughly analysed from a theoretical viewpoint by Grad (1967), who showed that non-pathological MHD equilibria are unlikely to exist in the absence of axial symmetry. This is connected, as is well known, to the constraint $\oint \mathrm{d}s/B =$ constant which has to be imposed on rational surfaces, this being unlikely to be possible for a low β plasma. Stellarators are typical examples of non-axisymmetric configurations, while tokamaks are, in principle, axisymmetric; the finite number of toroidal field coils, however, gives rise to small deviations from this symmetry in tokamaks.

Moreover, we remark that these phenomena are most likely responsible for the lack of convergence, at relatively large β, which takes place when one applies the Spitzer's iterative procedure (Spitzer 1958) to calculate the self-consistent magnetic field.

The need to amend the ideal MHD model, especially if the domain is lacking in symmetry, seems therefore well grounded and of significance for the study of plasmas confined by means of a magnetic field. For this purpose, the extensive literature concerning the theory of dissipative flow is of great relevance. In fact, on the basis of the mathematical theory of viscous incompressible flow (Ladyzhenskaya 1963, Temam 1979), one can conjecture that the mathematical pathologies highlighted by Grad can be due to the ideal character of the model which he analysed, and that the lack of symmetry of the domain can affect the shape of the equilibrium, but not preclude the existence of an equilibrium.

From a physical viewpoint, one can expect that the account in an MHD model of dissipative terms leads to a smoothing of all mathematical singularities. Moreover, the account of any force depending upon the plasma flow velocity and, in general, non-perpendicular to the magnetic field (e.g. the inertial force, the viscous force, the frictional force) leads to a decoupling of the magnetic surfaces from the pressure surfaces, the magnetic field being no longer constrained to be normal to the pressure gradient.

A dissipative model of plasma equilibrium, which obviously requires the presence of source terms in order to sustain the pressure gradient, was already addressed by Kruskal and Kulsrud (1958) who heuristically proved existence and uniqueness of solutions in the limiting case of low pressure. More recently, a dissipative model including resistivity and friction, but disregarding inertia and viscosity, was addressed (Wobig 1986). In this paper, we analyse a model whose equations include inertial forces, finite resistivity and viscosity, and a plasma source, and address the question of existence and uniqueness of solutions. The analysis is founded on the classical mathematical techniques adopted in the theory of viscous incompressible flow to investigate the solvability of the steady-state Navier–Stokes equations.

As the problem is nonlinear, the question of uniqueness is of no less significance than that of existence; here, we only derive a sufficient condition for uniqueness, and defer a more extensive analysis of bifurcation phenomena for this model to future work.

This paper is organized as follows. Section 2 contains an account of the model and the formulation of a boundary-value problem whose unknowns are the scalar pressure, the flow velocity and the magnetic field. Section 3 is concerned with the functional setting of the equations; suitable spaces of functions are introduced and a problem for weak solutions is established generalizing the techniques of mathematical hydrodynamics. In section 4, we show that the weak problem reduces to solving a nonlinear equation in a separable Hilbert space and, by analysing the Braginskii viscosity in the established functional framework, we find properties which allow this equation to be written as a fixed-point equation; the study of the Braginskii viscosity yields results of straightforward physical significance. By applying the Leray–Schauder principle (Gilbarg and Trudinger 1983), we obtain a condition under which the fixed-point equation has at least one solution, for which we obtain an estimate. Section 5 is concerned with the uniqueness of the solution; we prove that it holds under a condition of the same kind as that for existence, but which is more stringent. In section 6 we concisely summarize our main results and point out the questions that seem to deserve further consideration. Finally, in appendix A the well known existence and uniqueness results for the steady-state Navier–Stokes problem are recovered setting the magnetic field equal to zero, and in appendix B some non-trivial calculations are elucidated.

2. The model

We assume that the equilibrium of a plasma, filling a toroidal region Ω of the space $\mathbb{R}^3$, can be described by the following set of one-fluid, dissipative MHD equations:

$$\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \boldsymbol{j}\times\boldsymbol{B} + \hat{V}\boldsymbol{v} \tag{1}$$

$$\eta\boldsymbol{j} = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} \tag{2}$$

$$\boldsymbol{j} = \nabla\times\boldsymbol{B} \tag{3}$$

$$\nabla\cdot(\rho\boldsymbol{v}) = S \tag{4}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{5}$$

Here, ρ is the plasma density, η the resistivity, $\boldsymbol{v}$ the flow velocity, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{j}$ the current density, p the scalar pressure, $\boldsymbol{E}$ $(= -\nabla\phi)$ the electric field, S a particle source which sustains the pressure gradient in the plasma. Moreover, $\hat{V}\boldsymbol{v}$ is the Braginskii viscous force field (Braginskii 1965) given by $(\hat{V}\boldsymbol{v})_i = -\partial\pi_{ij}/\partial x_j$, $\pi_{ij} = \sum_{\alpha=0}^{4}\gamma_\alpha\mu_\alpha W_{\alpha ij}$ ($\gamma_\alpha \equiv -1$ for $\alpha = 0, 1, 2$ and $\gamma_\alpha \equiv 1$ for $\alpha = 3, 4$) where $W_{\alpha ij} = A_{\alpha ij,kl}(\boldsymbol{h})W_{kl}$ (repeated indices are summed); here, $\boldsymbol{h} \equiv \boldsymbol{B}/|\boldsymbol{B}|$ and W_{kl} is the rate-of-strain tensor: $W_{kl} = \partial_l v_k + \partial_k v_l - \frac{2}{3}\delta_{kl}\nabla\cdot\boldsymbol{v}$ $(= W_{lk})$; the coefficients $A_{\alpha ij,kl}$, which are polynomials in $\boldsymbol{h}$, are given on page 250 of Braginskii (1965).

The viscosity coefficients μ_α $(\alpha = 0, \ldots, 4)$ are positive and depend on $|\boldsymbol{B}|$ via $\omega\tau$, where ω is the gyrofrequency and τ is the collision time. The coefficient μ_0 describes the bulk viscosity, μ_1 and μ_2 the shear viscosity, μ_3 and μ_4 the gyroviscosity. Also, μ_0 is independent of the magnetic field, while μ_γ $(\gamma = 1, \ldots, 4)$ has an upper bound which is independent of the magnetic field. Shear viscosity and gyroviscosity tend to zero as $|\boldsymbol{B}| \to \infty$. In order to exclude this pathological situation, we introduce the approximation $\boldsymbol{B} \approx \boldsymbol{B}_0$ in the viscosity terms, $\boldsymbol{B}_0$ being the external vacuum field whose precise definition will be given later on. Since $|\boldsymbol{B}_0|$ is bounded in $\overline{\Omega}$, the viscosity coefficients μ_α have a lower bound: $\mu_\alpha \geqslant \overline{\mu} > 0$, for all $\alpha = 0, \ldots, 4$. Physically speaking, this implies that any flow velocity field (except the vanishing one) leads to dissipation of energy by the viscous forces. Note that, because of this approximation, the viscosity operator $\hat{V}$ is independent of the unknown magnetic field.

The system (1)–(5) is incomplete, since the equation of state correlating the density ρ and the pressure p is missing. The model which will be analysed in this paper is that of a uniform density, $\rho =$ constant. With $p \propto \rho T$ (where T denotes the temperature), the pressure gradient is proportional to the temperature gradient. Such a model is also supported by experimental results in stellarators, where very flat density profiles and peaked temperature profiles are found in electron cyclotron heated plasmas. It is this approximation which allows the system (1)–(5) to be reduced to the equations of incompressible fluid dynamics and to make use of the mathematical techniques developed in that field.

A complete description of the equilibrium would include an energy equation for the temperature T, with a given energy source Q and boundary conditions. In this case, the density would be, in general, inhomogeneous and should be determined by equations (1) and (4). In the model with a uniform density which we analyse in this paper the temperature is determined by the pressure; the energy equation may then be used to calculate the energy source Q which is needed to sustain the temperature profile. A weak point of this model is that boundary conditions on T must be ignored.

The resistivity η is a function of temperature, but we neglect this dependence and consider η also to be uniform. Similarly, the viscosity coefficients μ_α ($\alpha = 0, \ldots, 4$) are approximated by constants. At the end of section 4 we shall discuss the influence on our results of the assumption of uniform plasma density, resistivity and viscosity coefficients.

Moreover, as far as the electric field is concerned, it is, in general, described by a multivalued potential ϕ containing the toroidal loop voltage; therefore, the model is applicable to tokamak equilibria with flat density profiles too. However, in the first part of the analysis we consider the case without loop voltage; later on, it will be shown how the results are modified by a finite loop voltage.

We proceed by reducing the system (1)–(5) to a problem with unknowns $p, \boldsymbol{v}, \boldsymbol{B}$; let us use equation (3) in (1) and (2), and take the curl of equation (2). Thus, we obtain

$$\rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \hat{V}\boldsymbol{v} \tag{6}$$

$$\eta\nabla \times (\nabla \times \boldsymbol{B}) = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{7}$$

$$\rho\nabla \cdot \boldsymbol{v} = S \tag{8}$$

and $\nabla \cdot \boldsymbol{B} = 0$.

We supplement the system (6)–(8) with the following boundary conditions:

$$\boldsymbol{v} = \boldsymbol{v}_0 \qquad \text{on } \Gamma \tag{9}$$

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \qquad \text{and} \qquad \eta(\nabla \times \boldsymbol{B}) \times \boldsymbol{n} = (\boldsymbol{v}_0 \cdot \boldsymbol{n})\boldsymbol{B} \qquad \text{on } \Gamma \tag{10}$$

where $\Gamma = \partial\Omega$ is the boundary of Ω and $\boldsymbol{n}$ is the unit outward normal on Γ. The second condition of equation (10) expresses the requirement that the tangential component of $\boldsymbol{E}$ vanishes on Γ (the boundary is assumed to be a perfectly conducting wall).

We assume that Ω is a toroidal domain (i.e. an open connected set) of $\mathbb{R}^3$, and that the boundary Γ is a manifold of class C^∞. Moreover, we assume that Ω is Lipschitz (Marti 1986), so that it is regular enough to apply the Rellich–Kondrachov theorem later on (see section 4). Concerning S and $\boldsymbol{v}_0$, they are assumed to be smooth ($S \in C^\infty(\overline{\Omega})$ and $\boldsymbol{v}_0 \in (C^\infty(\Gamma))^3$; $\overline{\Omega}$ is the closure of Ω) and to fulfil the compatibility condition $\rho \int_\Gamma \mathrm{d}\sigma\, \boldsymbol{v}_0 \cdot \boldsymbol{n} = \int_\Omega \mathrm{d}^3x\, S(\boldsymbol{x})$.

The domain Ω is not simply connected; specifically, it is doubly connected. Problem (6)–(10) becomes well posed by prescribing the value of the toroidal flux of $\boldsymbol{B}$ (Sermange and Temam 1983, Foias and Temam 1978). Let $\boldsymbol{B}_0 \in (C^\infty(\overline{\Omega}))^3$ be the field having the prescribed toroidal flux, and fulfilling the following equations:

$$\begin{aligned} &\nabla \cdot \boldsymbol{B}_0 = 0 \qquad \text{and} \qquad \nabla \times \boldsymbol{B}_0 = \boldsymbol{0} \qquad \text{in } \Omega \\ &\boldsymbol{B}_0 \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma . \end{aligned} \tag{11}$$

Because of the topology of Ω, problem (11) has non-trivial solutions. Now we set

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_p \tag{12}$$

the field $\boldsymbol{B}_\mathrm{p}$ being our new unknown. In the following we shall omit the subscript p.

As regards the flow velocity field, let $\boldsymbol{v}_S \in (C^\infty(\overline{\Omega}))^3$ be one of the solutions of the following problem:

$$\begin{aligned} &\rho \nabla \cdot \boldsymbol{v}_S = S \qquad \text{in } \Omega \\ &\boldsymbol{v}_S = \boldsymbol{v}_0 \qquad \text{on } \Gamma . \end{aligned} \tag{13}$$

In the following we shall consider $\boldsymbol{v}_S$ as *given* and *fixed*. Setting

$$\boldsymbol{v} = \boldsymbol{v}_S + \boldsymbol{u} \tag{14}$$

the field $\boldsymbol{u}$ becomes our new unknown.

Next, we use equations (12) and (14) into the system (6)–(10); thus, for the unknowns p, $\boldsymbol{u}$ and $\boldsymbol{B}$ we have the following problem:

$$\begin{aligned} \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{v}_S + \rho(\boldsymbol{v}_S \cdot \nabla)\boldsymbol{u} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}_0 - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} \\ + \nabla(p + \tfrac{1}{2}|\boldsymbol{B}_0 + \boldsymbol{B}|^2) - \hat{V}\boldsymbol{u} = \boldsymbol{f}_S + (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{B}_0 \end{aligned} \tag{15}$$

$$\begin{aligned} \eta \nabla \times (\nabla \times \boldsymbol{B}) + \frac{S}{\rho}(\boldsymbol{B}_0 + \boldsymbol{B}) - (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{v}_S - (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}_S - (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{u} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} \\ + (\boldsymbol{v}_S \cdot \nabla)\boldsymbol{B}_0 + (\boldsymbol{v}_S \cdot \nabla)\boldsymbol{B} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B}_0 + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = \boldsymbol{0} \end{aligned} \tag{16}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{17}$$

This system is supplemented with the following boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma \tag{18}$$

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \qquad \text{and} \qquad \eta(\nabla \times \boldsymbol{B}) \times \boldsymbol{n} = (\boldsymbol{v}_S \cdot \boldsymbol{n})(\boldsymbol{B}_0 + \boldsymbol{B}) \qquad \text{on } \Gamma . \tag{19}$$

Here, well known identities as well as equations (5) and (8) have been used. Moreover, the field $\boldsymbol{f}_S$ appearing in equation (15) is defined by

$$\boldsymbol{f}_S \equiv -\rho(\boldsymbol{v}_S \cdot \nabla)\boldsymbol{v}_S + \hat{V}\boldsymbol{v}_S . \tag{20}$$

Note that $\boldsymbol{f}_S$ is a *given* quantity. It will play the role of an external force field.

3. Functional setting of the equations

Let $L^2(\Omega)$ be the space of real-valued functions on Ω which are square integrable for the Lebesgue measure $\mathrm{d}^3x = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$; this is a Hilbert space for the scalar product $(\xi, \xi') = \int_\Omega \mathrm{d}^3x\, \xi(\boldsymbol{x})\xi'(\boldsymbol{x})$. Let $H^m(\Omega)$ be the Sobolev space of functions which are in $L^2(\Omega)$ together with their weak derivatives of order less than or equal to m (Adams 1975); $H_0^m(\Omega)$ is the Hilbert subspace of $H^m(\Omega)$ made of functions vanishing on Γ. Moreover, we use the notation $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^3$, $\boldsymbol{H}^m(\Omega) = (H^m(\Omega))^3$, $\boldsymbol{H}_0^m(\Omega) = (H_0^m(\Omega))^3$.

We shall use the following spaces:

$$
\begin{aligned}
&\mathcal{V}_1 = \{\boldsymbol{v} \in (\mathcal{C}_c^\infty(\Omega))^3, \nabla\cdot\boldsymbol{v} = 0\} \\
&V_1 = \text{the closure of } \mathcal{V}_1 \text{ in } \boldsymbol{H}_0^1(\Omega) \\
&\mathcal{V}_2 = \left\{\boldsymbol{B} \in (\mathcal{C}^\infty(\overline{\Omega}))^3, \nabla\cdot\boldsymbol{B} = 0, \boldsymbol{B}\cdot\boldsymbol{n}|_\Gamma = 0 \text{ and } \int_\Sigma \mathrm{d}\sigma\, \boldsymbol{B}\cdot\boldsymbol{n} = 0\right\} \\
&V_2 = \text{the closure of } \mathcal{V}_2 \text{ in } \boldsymbol{H}^1(\Omega)
\end{aligned}
\tag{21}
$$

where Σ is any smooth manifold of dimension two such that the open set $\Omega \setminus \Sigma$ is simply-connected and Lipschitz (i.e. Σ is not tangent to Γ); roughly speaking, Σ is a poloidal cut.

We equip V_1 with the scalar product

$$((\boldsymbol{v}, \boldsymbol{v}'))_1 = (\partial_i \boldsymbol{v}, \partial_i \boldsymbol{v}') \tag{22}$$

where $\partial_i = \partial/\partial x_i$ and, as always, repeated indices are summed. This is a scalar product on $\boldsymbol{H}_0^1(\Omega)$ thanks to the Poincaré inequality, and provides the norm on V_1 given by $\|\boldsymbol{v}\|_1 = \{((\boldsymbol{v}, \boldsymbol{v}))_1\}^{1/2}$.

We equip V_2 with the scalar product (Sermange and Temam 1983)

$$((\boldsymbol{B}, \boldsymbol{B}'))_2 = (\nabla\times\boldsymbol{B}, \nabla\times\boldsymbol{B}'). \tag{23}$$

The topology of Ω is here of fundamental importance: since Ω is doubly connected, this bilinear form is actually a scalar product on V_2 only if (see equation (21)) the constraint of zero toroidal flux is imposed. Such a very technical result can be deduced from the theorems proved in Foias and Temam (1978). The scalar product (23) defines a norm on V_2 given by $\|\boldsymbol{B}\|_2 = \{((\boldsymbol{B}, \boldsymbol{B}))_2\}^{1/2}$, which is equivalent to that induced by $\boldsymbol{H}^1(\Omega)$ on V_2; see Sermange and Temam (1983).

Finally, we introduce the product space

$$V = V_1 \times V_2 \tag{24}$$

and equip it with the scalar product

$$((\Phi, \Phi')) = \mu_\star((\boldsymbol{v}, \boldsymbol{v}'))_1 + \eta((\boldsymbol{B}, \boldsymbol{B}'))_2 \qquad \text{for all } \Phi = (\boldsymbol{v}, \boldsymbol{B}),\ \Phi' = (\boldsymbol{v}', \boldsymbol{B}') \in V \tag{25}$$

where $\mu_\star \equiv \frac{1}{3}\min_{\alpha=0,1,2} \mu_\alpha$. This scalar product provides the norm on V given by $\|\Phi\| = \{((\Phi, \Phi))\}^{1/2}$.

We proceed now by establishing a weak formulation of problem (15)–(19).

Let us assume that p, $\boldsymbol{u}$, $\boldsymbol{B}$ is a *smooth solution*. The first step is to multiply equation (15) by a test function $\boldsymbol{w} \in V_1$ and integrate over Ω. Note that, for all $\zeta \in C^\infty(\overline{\Omega})$, we have

$$\int_\Omega \mathrm{d}^3x\,(\nabla\zeta)\cdot\boldsymbol{w} = \int_\Omega \mathrm{d}^3x\,[\nabla\cdot(\zeta\boldsymbol{w}) - \zeta\nabla\cdot\boldsymbol{w}] = \int_\Gamma \mathrm{d}\sigma\,\zeta\boldsymbol{w}\cdot\boldsymbol{n} = 0 \tag{26}$$

and also

$$\int_\Omega \mathrm{d}^3x\,[(\boldsymbol{B}_0\cdot\nabla)\boldsymbol{B}_0]\cdot\boldsymbol{w} = \int_\Omega \mathrm{d}^3x\,[(\nabla\times\boldsymbol{B}_0)\times\boldsymbol{B}_0 + \nabla(\tfrac{1}{2}|\boldsymbol{B}_0|^2)]\cdot\boldsymbol{w} = 0 \tag{27}$$

where we have used equations (11) and (26). Concerning the quantity $(-\hat{V}\boldsymbol{u},\boldsymbol{w})$ arising from the left-hand side of equation (15), we proceed in the following way: let us introduce the following bilinear form

$$\begin{aligned}
&\mathcal{E}\colon V_1\times V_1 \to \mathbb{R}\\
&(\boldsymbol{a},\boldsymbol{b}) \mapsto \mathcal{E}(\boldsymbol{a},\boldsymbol{b})\\
&\mathcal{E}(\boldsymbol{a},\boldsymbol{b}) \equiv -\sum_{\alpha=0}^{4}\gamma_\alpha\mu_\alpha\int_\Omega \mathrm{d}^3x\,\partial_l a_k(A_{\alpha ij,kl} + A_{\alpha ij,lk})\partial_j b_i.
\end{aligned} \tag{28}$$

One can easily check that, since $\boldsymbol{u}$ is assumed to be a smooth solution and $\boldsymbol{w}$ to belong to V_1, the identity $(-\hat{V}\boldsymbol{u},\boldsymbol{w}) = \mathcal{E}(\boldsymbol{u},\boldsymbol{w})$ holds. Moreover, by using trivial inequalities as well as the Cauchy–Schwarz inequality for sums and for integrals, we can easily convince ourselves that, $\forall\boldsymbol{a}\in V_1$ fixed, the mapping

$$\begin{aligned}
&\mathcal{E}(\boldsymbol{a},\bullet)\colon V_1 \to \mathbb{R}\\
&\boldsymbol{b} \mapsto \mathcal{E}(\boldsymbol{a},\boldsymbol{b})
\end{aligned} \tag{29}$$

is a *bounded* linear functional (we recall that the coefficients $A_{\alpha ij,kl}$ are polynomials in $\boldsymbol{h}$). Therefore, by using the Riesz representation theorem, we see that there exists one and only one $\tilde{\boldsymbol{a}} \in V_1$ such that $\mathcal{E}(\boldsymbol{a},\boldsymbol{b}) = ((\tilde{\boldsymbol{a}},\boldsymbol{b}))_1 \forall\boldsymbol{b}\in V_1$. Since, for $\boldsymbol{a}\in V_1$ fixed, the element $\tilde{\boldsymbol{a}}\in V_1$ is unique, we can give the following good definition of the operator $\tilde{E}$:

$$\begin{aligned}
&\tilde{E}\colon V_1' \to V_1\\
&\boldsymbol{a} \mapsto \tilde{E}\boldsymbol{a} \equiv \tilde{\boldsymbol{a}}.
\end{aligned} \tag{30}$$

It is also advantageous to introduce the operator E by setting $\tilde{E} \equiv \mu_\star E$ so that, finally, we have

$$(-\hat{V}\boldsymbol{u},\boldsymbol{w}) = \mu_\star((E\boldsymbol{u},\boldsymbol{w}))_1. \tag{31}$$

Note that the operator $\tilde{E}$ is linear (and, hence, the operator E), as $\mathcal{E}$ is a bilinear form.

In order to shorten the notation, we introduce a trilinear form on $(\boldsymbol{H}^1(\Omega))^3$ by setting

$$b(\xi,\xi',\xi'') = \int_\Omega \mathrm{d}^3x\, \xi_i(\partial_i \xi_j')\xi_j''. \tag{32}$$

This form is *continuous* (Sermange and Temam 1983).

Thus, by using equations (26)–(27) and (31) as well as definition (32), the previously mentioned projection of equation (15) yields the following (weak) equation:

$$\begin{aligned}\mu_*((E\boldsymbol{u},\boldsymbol{w}))_1 &+ \rho b(\boldsymbol{u},\boldsymbol{v}_S,\boldsymbol{w}) + \rho b(\boldsymbol{v}_S,\boldsymbol{u},\boldsymbol{w}) + \rho b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) \\ &- b(\boldsymbol{B}_0,\boldsymbol{B},\boldsymbol{w}) - b(\boldsymbol{B},\boldsymbol{B}_0,\boldsymbol{w}) - b(\boldsymbol{B},\boldsymbol{B},\boldsymbol{w}) = (\boldsymbol{f}_S,\boldsymbol{w}).\end{aligned} \tag{33}$$

Note that the right-hand side of equation (33) makes sense because, under our hypotheses, we have that $\boldsymbol{f}_S \in (C^\infty(\overline{\Omega}))^3$.

Next, let us deal with equation (16) and remember we are assuming p, $\boldsymbol{u}$, $\boldsymbol{B}$ to be a smooth solution. We proceed in the following way (see also Sermange and Temam (1983)): we multiply equation (16) by a test function $\boldsymbol{C} \in V_2$ and integrate over Ω. Note that the identity

$$\int_\Omega \mathrm{d}^3x\,[\nabla\times(\nabla\times\boldsymbol{B})]\cdot\boldsymbol{C} = \int_\Omega \mathrm{d}^3x\,(\nabla\times\boldsymbol{B})\cdot(\nabla\times\boldsymbol{C}) - \int_\Gamma \mathrm{d}\sigma\,[(\nabla\times\boldsymbol{B})\times\boldsymbol{n}]\cdot\boldsymbol{C}$$

holds. In its last term we use equation (19); moreover, performing some integrations by parts in the projection of equation (16) we see that several cancellations take place. As a result of this straightforward calculation, we obtain the following (weak) equation:

$$\begin{aligned}\eta((\boldsymbol{B},\boldsymbol{C}))_2 &+ b(\boldsymbol{u},\boldsymbol{B}_0+\boldsymbol{B},\boldsymbol{C}) - b(\boldsymbol{B}_0+\boldsymbol{B},\boldsymbol{u},\boldsymbol{C}) \\ &- b(\boldsymbol{B}_0+\boldsymbol{B},\boldsymbol{v}_S,\boldsymbol{C}) - b(\boldsymbol{v}_S,\boldsymbol{C},\boldsymbol{B}_0+\boldsymbol{B}) = 0\end{aligned} \tag{34}$$

where we have used equation (23).

In order to establish a problem for weak solutions in the product space V, we introduce the following operator:

$$\begin{aligned}&U\colon V\to V \\ &\Phi = (\boldsymbol{v},\boldsymbol{B}) \mapsto U\Phi \equiv (E\boldsymbol{v},\boldsymbol{B}).\end{aligned} \tag{35}$$

Note that U is a linear operator as E is linear.

Furthermore, in order to shorten the notation, let us define the following mapping:

$$\mathcal{B}\colon V\times V\to\mathbb{R}$$

$$(\Phi,\Phi')\mapsto \mathcal{B}(\Phi,\Phi')$$

$$\begin{aligned}\mathcal{B}(\Phi,\Phi') \equiv\ &\rho b(\boldsymbol{v},\boldsymbol{v}_S,\boldsymbol{v}') + \rho b(\boldsymbol{v}_S,\boldsymbol{v},\boldsymbol{v}') + \rho b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{v}') - b(\boldsymbol{B}_0,\boldsymbol{B},\boldsymbol{v}') - b(\boldsymbol{B},\boldsymbol{B}_0,\boldsymbol{v}') \\ &- b(\boldsymbol{B},\boldsymbol{B},\boldsymbol{v}') + b(\boldsymbol{v},\boldsymbol{B}_0+\boldsymbol{B},\boldsymbol{B}') - b(\boldsymbol{B}_0+\boldsymbol{B},\boldsymbol{v},\boldsymbol{B}') \\ &- b(\boldsymbol{B}_0+\boldsymbol{B},\boldsymbol{v}_S,\boldsymbol{B}') - b(\boldsymbol{v}_S,\boldsymbol{B}',\boldsymbol{B}_0+\boldsymbol{B})\end{aligned} \tag{36}$$

where $\Phi = (\boldsymbol{v},\boldsymbol{B})$ and $\Phi' = (\boldsymbol{v}',\boldsymbol{B}')$. Note that the mapping $\mathcal{B}$ is manifestly linear in the second argument but nonlinear in the first one.

Now, we add equations (33) and (34) and use equations (35), (25), (36); thus, we obtain the following (weak) equation:

$$((U\Phi,\Psi)) + \mathcal{B}(\Phi,\Psi) = (\boldsymbol{f}_S,\boldsymbol{w}) \tag{37}$$

where $\Phi = (\boldsymbol{u},\boldsymbol{B})$ and $\Psi = (\boldsymbol{w},\boldsymbol{C})$.

We can now establish the following weak formulation of problem (15)–(19):

Problem (weak solutions). Under the previous hypotheses for Ω, $\boldsymbol{v}_S$ and $\boldsymbol{B}_0$, find $\Phi = (\boldsymbol{u}, \boldsymbol{B}) \in V$ such that equation (37) is satisfied for all $\Psi = (\boldsymbol{w}, \boldsymbol{C}) \in V$.

Note that we do not require the solution to be smooth, since we look for it in $V = V_1 \times V_2$ and not in $\mathcal{V}_1 \times \mathcal{V}_2$. For a thorough discussion on the weak formulation of problems of this kind see Ladyzhenskaya (1963), Temam (1979) and Sermange and Temam (1983). We only remark here that it is not obvious at all how the second condition of equation (19) is recovered; as far as this point is concerned, see Duvaut and Lions (1972).

4. Existence and estimate of weak solutions

We proceed considering the question of existence of the previously defined weak solutions. As we shall see, proving existence also yields an estimate of the solution(s). The mathematical techniques we are going to use are classical for problems of this kind (Ladyzhenskaya 1963, Temam 1979); nevertheless, since we describe viscosity by the Braginskii operator while in previous work the Laplace operator was always used, we shall have to carry out a special analysis with respect to this point.

To investigate the solvability of the weak problem we established earlier, we are going to take the following steps:

(i) formulate the problem in terms of solvability of a (nonlinear) equation in the product space V;

(ii) write this equation as a fixed-point equation; and

(iii) investigate the solvability of this fixed-point equation by using a theorem which yields existence but not uniqueness.

First, we consider the right-hand side of equation (37). We have trivially that $|(\boldsymbol{f}_S, \boldsymbol{w})| \leqslant \|\boldsymbol{f}_S\|_{L^2(\Omega)} \|\boldsymbol{w}\|_{L^2(\Omega)} \leqslant \|\boldsymbol{f}_S\|_{L^2(\Omega)} \|\boldsymbol{w}\|_{H_0^1(\Omega)}$; moreover, as the norm $\|\bullet\|_1$ is equivalent to the norm $\|\bullet\|_{H_0^1(\Omega)}$ (Temam 1979), the last quantity is less than or equal to a positive constant times $\|\boldsymbol{f}_S\|_{L^2(\Omega)} \|\boldsymbol{w}\|_1$. Therefore, the mapping

$$
\begin{aligned}
&V \to \mathbb{R} \\
&\Psi = (\boldsymbol{w}, \boldsymbol{C}) \mapsto (\boldsymbol{f}_S, \boldsymbol{w})
\end{aligned}
\tag{38}
$$

is a bounded linear functional (note that, from equation (25), it follows trivially that $\|\boldsymbol{w}\|_1 \leqslant \|\Psi\| / \sqrt{\mu_*}$ and $\|\boldsymbol{C}\|_2 \leqslant \|\Psi\| / \sqrt{\eta}$). According to Riesz's theorem, the functional (38) can be represented in the form $(\boldsymbol{f}_S, \boldsymbol{w}) = ((F_S, \Psi))$ for one and only one element $F_S \in V$; clearly, the second component of F_S is equal to zero.

Next, we consider the term $\mathcal{B}(\Phi, \Psi)$ in the left-hand side of equation (37). This quantity (see definition (36)) is a linear combination of b-forms; each b-form has, among its arguments, either $\boldsymbol{w}$ or $\boldsymbol{C}$ ($\boldsymbol{w}$ and $\boldsymbol{C}$ never appearing together). As the trilinear form b is continuous on $(\boldsymbol{H}^1(\Omega))^3$, the estimate $|\mathcal{B}(\Phi, \Psi)| \leqslant \sum_{\tau=1}^{10} c_\tau \|\boldsymbol{a}_\tau\|_{H^1(\Omega)}$ clearly holds; here, c_τ are non-negative quantities which do not depend on Ψ, and $\boldsymbol{a}_\tau$ is either $\boldsymbol{w}$ or $\boldsymbol{C}$. Since, as we said, the norms $\|\bullet\|_1$ and $\|\bullet\|_2$ are equivalent to the norm $\|\bullet\|_{H^1(\Omega)}$, this estimate also holds with c_τ replaced by other constants c'_τ and the norm in $\boldsymbol{H}^1(\Omega)$ replaced by $\|\bullet\|_1$ (for $\boldsymbol{w}$) and $\|\bullet\|_2$ (for $\boldsymbol{C}$). Finally, remembering the remark which follows equation (38), this estimate also holds with c_T' replaced by other constants c_T'' and the norms $||w||_1$ and $||C||_2$ replaced by $||\Psi||$. Hence, we conclude that, $\forall \Phi \in V$ fixed, the mapping

$$\begin{aligned} &B(\Phi,\bullet): V \to \mathbb{R} \\ &\Psi \mapsto B(\Phi,\Psi) \end{aligned} \tag{39}$$

is a bounded linear functional. Proceeding as after equation (29), we see that there exists an operator $\tilde{B}: V \to V$ such that $B(\Phi,\Psi) = ((\tilde{B}\Phi,\Psi))$. The operator $\tilde{B}$ is clearly nonlinear.

Thus, going back to equation (37), we can write it in the following way:

$$((U\Phi,\Psi)) + ((\tilde{B}\Phi,\Psi)) = ((F_S,\Psi)). \tag{40}$$

It is advantageous to introduce the *constant operator* $C_S: V \to V$ *such that* $\Phi \mapsto C_S\Phi \equiv F_S$ for all $\Phi \in V$, and also the operator $Z \equiv C_S - \tilde{B}$.

The element $\Phi \in V$ is a weak solution of our problem if and only if equation (40) is satisfied for all $\Psi \in V$. Therefore, the weak problem reduces to solving the nonlinear equation

$$U\Phi = Z\Phi \tag{41}$$

in the space V.

The next step is to prove that the operator U is one-to-one, so that equation (41) can be written as a fixed-point equation. Hence, in the following part of the analysis we master the properties of U.

As we already remarked, the operator U is linear. Therefore, the mapping

$$\begin{aligned} &V \times V \to \mathbb{R} \\ &(\Phi,\Psi) \mapsto ((U\Phi,\Psi)) \end{aligned} \tag{42}$$

is a bilinear form. Moreover, it is bounded; this property is inherited from the bilinear form $\mathcal{E}$ defined by equation (28). In fact, in relation to the mapping (29), we noted that $\mathcal{E}$ is bounded *in the second argument*; we can easily convince ourselves that this holds for both arguments, so that a constant e exists such that $|\mathcal{E}(a,b)| \leqslant e||a||_1||b||_1$ for all $a,b \in V_1$. Thus, we can write the following estimates:

$$\begin{aligned} |((U\Phi,\Psi))| &\leqslant ||U\Phi||||\Psi|| = \sqrt{\mu_*||Ev||_1^2 + \eta||B||_2^2}||\Psi|| = \sqrt{\mathcal{E}(v,Ev) + \eta||B||_2^2}||\Psi|| \\ &\leqslant \sqrt{e||v||_1||Ev||_1 + \eta||B||_2^2}||\Psi|| \leqslant \sqrt{\frac{e^2}{\mu_*}||v||_1^2 + \eta||B||_2^2}||\Psi|| \\ &\leqslant \max\left\{1,\frac{e}{\mu_*}\right\}||\Phi||||\Psi|| \end{aligned} \tag{43}$$

where $\Phi = (v,B)$. Equation (43) shows that the bilinear form (42) is bounded.

Next, we prove that the form (42) has another interesting property: it is coercive. As before, such a property is inherited from the form $\mathcal{E}$. First, note that $((U\Phi,\Phi)) = \mathcal{E}(v,v) + \eta||B||_2^2$. As regards the form $\mathcal{E}$ with equal arguments, we are going to write a chain of estimates from below which manifestly hold if we notice that:

(i) The tensors $W_{\alpha ij}$ ($\alpha = 0,\ldots,4$) are symmetric, since π_{ij} is symmetric (Braginskii 1965);

(ii) $W_{ij} = \sum_{\alpha=0}^{2} W_{\alpha ij}$ (Braginskii 1965);

(iii) $W_{\alpha ij} W_{\beta ij} = 0$ when $\alpha \neq \beta$ (Braginskii 1965);

(iv) $\sum_{\alpha=0}^{n} \xi_\alpha \leqslant \sqrt{n+1}(\sum_{\alpha=0}^{n} \xi_\alpha^2)^{1/2}$ for all $(n+1)$-tuples $(\xi_0,\ldots,\xi_n) \in \mathbb{R}^{n+1}$ (Cauchy–Schwarz inequality);

(v) $\boldsymbol{v} = \boldsymbol{0}$ on Γ and $\nabla \cdot \boldsymbol{v} = 0$ in Ω.

Moreover, setting $G_{\alpha ij,kl} \equiv A_{\alpha ij,kl} + A_{\alpha ij,lk}$ and carrying out a somewhat long algebraic analysis, we can see that $G_{\alpha ij,kl} = G_{\alpha kl,ij}$ for $\alpha = 0,2$; $G_{\alpha ij,kl} = -G_{\alpha kl,ij}$ for $\alpha = 3,4$; and neither equality holds for $\alpha = 1$. Therefore, remembering equation (28) and setting $\mathcal{E} = \sum_{\alpha=0}^{4} \mathcal{E}_\alpha$, we have that $\mathcal{E}_0$ and $\mathcal{E}_2$ are symmetric, $\mathcal{E}_3$ and $\mathcal{E}_4$ are anti-symmetric, and $\mathcal{E}_1$ is neither symmetric nor anti-symmetric. The relevant consequence for us is that the terms $\alpha = 3,4$ give no contribution to $\mathcal{E}(\boldsymbol{v},\boldsymbol{v})$. By using, finally, all these properties, we can write the following chain of estimates from below:

$$\begin{aligned}
\mathcal{E}(\boldsymbol{v},\boldsymbol{v}) &= \sum_{\alpha=0}^{2} \mu_\alpha \int_\Omega \mathrm{d}^3x\, \partial_l v_k (A_{\alpha ij,kl} + A_{\alpha ij,lk}) \partial_j v_i \\
&= \sum_{\alpha=0}^{2} \mu_\alpha \int_\Omega \mathrm{d}^3x\, \partial_j v_i W_{\alpha ij} = \frac{1}{2} \sum_{\alpha=0}^{2} \mu_\alpha \int_\Omega \mathrm{d}^3x\, (\partial_j v_i + \partial_i v_j) W_{\alpha ij} \\
&= \frac{1}{2} \sum_{\alpha=0}^{2} \mu_\alpha \int_\Omega \mathrm{d}^3x\, W_{ij} W_{\alpha ij} = \frac{1}{2} \sum_{\alpha,\beta=0}^{2} \mu_\alpha \int_\Omega \mathrm{d}^3x\, W_{\beta ij} W_{\alpha ij} \\
&= \frac{1}{2} \sum_{\alpha=0}^{2} \mu_\alpha \int_\Omega \mathrm{d}^3x \sum_{i,j} W_{\alpha ij}^2 \geqslant \frac{3}{2} \mu_\star \sum_{i,j} \int_\Omega \mathrm{d}^3x \sum_{\alpha=0}^{2} W_{\alpha ij}^2 \\
&\geqslant \frac{1}{2} \mu_\star \sum_{i,j} \int_\Omega \mathrm{d}^3x \left(\sum_{\alpha=0}^{2} W_{\alpha ij} \right)^2 = \frac{1}{2} \mu_\star \sum_{i,j} \int_\Omega \mathrm{d}^3x\, W_{ij}^2 \\
&= \mu_\star \int_\Omega \mathrm{d}^3x \sum_{i,j} (\partial_i v_j)^2 = \mu_\star \|\boldsymbol{v}\|_1^2.
\end{aligned} \tag{44}$$

Equation (44) shows that the form $\mathcal{E}$ is coercive. This property has a plain and elegant physical interpretation: the viscous forces do always dissipate energy. In fact, we point out that, if $\boldsymbol{v}$ is smooth, $\mathcal{E}(\boldsymbol{v},\boldsymbol{v})$ is simply the power dissipated by the viscous forces in the domain Ω, and that from equation (44) it follows that $\mathcal{E}(\boldsymbol{v},\boldsymbol{v}) = 0$ implies $\boldsymbol{v} = \boldsymbol{0}$. This means, roughly speaking, that the Braginskii viscosity operator is negative definite; also, note that we have recovered a known property of the gyroviscosity (connected with $\alpha = 3,4$): it is non-dissipative.

The form (42) inherits this property, as we have

$$((U\Phi,\Phi)) = \mathcal{E}(\boldsymbol{v},\boldsymbol{v}) + \eta\|\boldsymbol{B}\|_2^2 \geqslant \mu_\star\|\boldsymbol{v}\|_1^2 + \eta\|\boldsymbol{B}\|_2^2 = \|\Phi\|^2. \tag{45}$$

Since the bilinear form (42) is bounded and coercive, we can deduce, as in the proof of the Lax–Milgram theorem (Gilbarg and Trudinger 1983), that the operator U is one-to-one and U^{-1} is bounded; moreover, we have that

$$\|U^{-1}\Phi\| \leqslant \|\Phi\| \leqslant \max\left\{1, \frac{e}{\mu_\star}\right\} \|U^{-1}\Phi\|. \tag{46}$$

Thus, we can go back to equation (41) and write it equivalently as a fixed-point equation:

$$U^{-1}Z\Phi = \Phi. \tag{47}$$

To investigate the solvability of equation (47), we apply the Leray–Schauder principle (Ladyzhenskaya 1963). This principle is particularly suitable for problems of this kind since it guarantees existence but not uniqueness.

The first step is to check that V, the space in which equation (47) is defined, is a separable Hilbert space (i.e. it has a countable dense subset). The space $H^1(\Omega)$ is a separable Hilbert space (Adams 1975). Remembering equation (24), that V_1 and V_2 are Hilbert subspaces of $H^1(\Omega)$, and that they are equipped with norms equivalent to the norm $\|\bullet\|_{H^1(\Omega)}$, we can immediately state that V is a separable Hilbert space.

The second step is to check that the operator $U^{-1}Z$ is completely continuous in V, i.e. it maps any weakly convergent sequence $\{\Phi_n\}$ in V into a strongly convergent sequence $\{U^{-1}Z\Phi_n\}$ in V. To prove that the operator at issue has such a property, we need, first, some information concerning imbeddings of Sobolev spaces. We recall that a normed space X is said to be imbedded in the normed space Y provided: (i) X is a vector subspace of Y, and (ii) the identity operator I defined on X into Y is continuous; we write $X \to Y$ to designate this imbedding. Condition (ii) is equivalent to the existence of a constant M such that $\|Ix\|_Y \leqslant M\|x\|_X$ for all $x \in X$. We say that X is compactly imbedded in Y if the imbedding operator I is compact. As regards our problem, from the Rellich–Kondrachov theorem (Adams 1975) it follows that, under our hypotheses, the compact imbedding $H^1(\Omega) \to L^q(\Omega)$ holds, with $1 \leqslant q < 6$. The relevant consequence for us is that, if $\{(u_n, B_n)\}$ is a weakly convergent sequence in V, then this sequence converges strongly in $L^4(\Omega) \times L^4(\Omega)$. (Note that the imbedding operator is continuous, by definition, and compact, so that it is completely continuous.)

In fact, going back to the complete continuity of the operator $U^{-1}Z$, we proceed in the following way. First, note that, since U^{-1} is linear and bounded, it is sufficient to prove that Z is completely continuous. For this purpose, let us consider an element $\Psi \in V$ and the quantity $((Z\Phi_m - Z\Phi_n, \Psi)) = \mathcal{B}(\Phi_n, \Psi) - \mathcal{B}(\Phi_m, \Psi)$. We must estimate the right-hand side of this equality; remembering definition (36), using (as always in this analysis) trivial inequalities and the Cauchy–Schwarz inequality for sums and for integrals, and devising plain artifices, we obtain after some long calculations that $|\mathcal{B}(\Phi_n, \Psi) - \mathcal{B}(\Phi_m, \Psi)| \leqslant \sum_\tau c_{nm}^{(\tau)}\|a_{nm}^{(\tau)}\|_{L^4(\Omega)}\|\Psi\|$; here, $c_{nm}^{(\tau)}$ remain bounded as $n, m \to \infty$ because of the strong convergence of the sequences $\{u_n\}$ and $\{B_n\}$ in $L^4(\Omega)$, and $a_{nm}^{(\tau)}$ is either $u_n - u_m$ or $B_n - B_m$. Therefore, setting $\Psi = Z\Phi_m - Z\Phi_n$, we have that $\|Z\Phi_m - Z\Phi_n\| \to 0$ as $n, m \to \infty$, namely the operator Z is completely continuous.

Since V is a separable Hilbert space and $U^{-1}Z$ is a completely continuous operator, the Leray–Schauder principle guarantees that, if all possible solutions of the equation $\lambda U^{-1}Z\Phi = \Phi$ for $\lambda \in [0, 1]$ lie within some ball $\|\Phi\| \leqslant R$, then the equation (47) has at least one solution inside this ball.

We proceed by noticing that

$$\begin{aligned} \|\Phi^{(\lambda)}\|^2 &\leqslant ((U\Phi^{(\lambda)}, \Phi^{(\lambda)})) = ((\lambda Z\Phi^{(\lambda)}, \Phi^{(\lambda)})) \\ &= \lambda((F_S, \Phi^{(\lambda)})) - \lambda\mathcal{B}(\Phi^{(\lambda)}, \Phi^{(\lambda)}) \\ &\leqslant \|F_S\|\|\Phi^{(\lambda)}\| + |\mathcal{B}(\Phi^{(\lambda)}, \Phi^{(\lambda)})| \end{aligned} \tag{48}$$

where we have used equation (45) and the fact that U is linear. Next, we must consider the mapping B with equal arguments. A careful analysis of this quantity shows that several terms annul each other; the result of this non-trivial calculation is the following (see appendix B):

$$\begin{aligned} B(\Phi^{(\lambda)}, \Phi^{(\lambda)}) &= \rho \int_\Omega \mathrm{d}^3x\, (\boldsymbol{v}_S \times \boldsymbol{u}^{(\lambda)}) \cdot (\nabla \times \boldsymbol{u}^{(\lambda)}) \\ &\quad + \int_\Omega \mathrm{d}^3x\, [(\boldsymbol{B}_0 + \boldsymbol{B}^{(\lambda)}) \times \boldsymbol{v}_S] \cdot (\nabla \times \boldsymbol{B}^{(\lambda)}). \end{aligned} \tag{49}$$

Now, we estimate suitably the right-hand side of equation (49). After a long calculation we obtain

$$\begin{aligned} |B(\Phi^{(\lambda)}, \Phi^{(\lambda)})| &\leqslant \|\Phi^{(\lambda)}\|^2 \|\boldsymbol{v}_S\|_{\boldsymbol{L}^4(\Omega)} \left(\frac{3\sqrt{3}\rho M_1}{\mu_\star} + \frac{M_2}{\eta} \right) \\ &\quad + \frac{1}{\sqrt{\eta}} \|\Phi^{(\lambda)}\| \|\boldsymbol{v}_S\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{B}_0\|_{\boldsymbol{L}^4(\Omega)}. \end{aligned} \tag{50}$$

Here, M_i ($i = 1, 2$) is the imbedding constant of the compact imbedding $V_i(\|\bullet\|_i) \rightarrow \boldsymbol{L}^4(\Omega)$; note that it depends only on Ω. Using equations (48) and (50) we obtain

$$\|\Phi^{(\lambda)}\| \left[1 - \|\boldsymbol{v}_S\|_{\boldsymbol{L}^4(\Omega)} \left(\frac{3\sqrt{3}\rho M_1}{\mu_\star} + \frac{M_2}{\eta} \right) \right] \leqslant \frac{1}{\sqrt{\eta}} \|\boldsymbol{v}_S\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{B}_0\|_{\boldsymbol{L}^4(\Omega)} + \|F_S\|. \tag{51}$$

From equation (51) it follows that, if

$$\|\boldsymbol{v}_S\|_{\boldsymbol{L}^4(\Omega)} \left(\frac{3\sqrt{3}\rho M_1}{\mu_\star} + \frac{M_2}{\eta} \right) < 1 \tag{52}$$

then the norms $\|\Phi^{(\lambda)}\|$ are uniformly bounded. Therefore, if the condition (52) is satisfied, at least one weak solution of our problem does exist. The requirement is that the source must be sufficiently small or the viscosity and resistivity sufficiently large.

Moreover, for the solution(s) the following estimate holds:

$$\|\Phi\| \leqslant \frac{(1/\sqrt{\eta}) \|\boldsymbol{v}_S\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{B}_0\|_{\boldsymbol{L}^4(\Omega)} + \|F_S\|}{1 - \|\boldsymbol{v}_S\|_{\boldsymbol{L}^4(\Omega)} ((3\sqrt{3}\rho M_1/\mu_\star) + (M_2/\eta))}. \tag{53}$$

As one could expect, this estimate shows that the larger the viscosity and resistivity are (or the smaller the source), the smaller $\|\Phi\|$ is: i.e. the dissipation quenches the flow velocity and the plasma currents. In particular, if the source vanishes we have that $\Phi = 0$, i.e. the flow velocity vanishes and no current flows in the plasma. (We recall that we assumed there is no loop voltage; in the presence of loop voltage, another positive quantity would appear in the numerator of equation (53) and a non-trivial solution could exist even if the source vanishes.)

As we have already remarked, the solution(s) whose existence we have proved may be non-smooth. As a matter of fact, they may have so little regularity as to be hardly considered significant from the point of view of physics. Nevertheless, we point out that, for the steady-state Navier–Stokes equations, C^∞-regularity of the domain and of the force field implies C^∞-regularity of the solution(s) (see Temam (1979) on page 172); it is clear that the same can be expected to hold for the model we are analysing here.

Equations (52)–(53), together with the condition for uniqueness we are going to derive and this study of the Braginskii viscosity operator, are the main results of this analysis.

In section 2 we briefly discussed the assumption of a given uniform plasma density. The generalization of the existence theorem to the case in which ρ is a further unknown is, as a matter of fact, a very difficult task: one should clearly generalize theorems which hold for viscous compressible flows, the mathematical theory of which is of very great complexity and still incomplete. The analysis of a model in which the dependence of the viscosity and resistivity coefficients upon the unknowns is taken into account is even more difficult; indeed, such an analysis does not seem feasible. On the contrary, we believe that the existence result presented here can be generalized to the case in which plasma density, resistivity and viscosity coefficients have a given (sufficiently regular) space dependence. Suppose that these quantities belong to the class $C^\infty(\overline{\Omega})$ and that, in $\overline{\Omega}$, they are greater than or equal to respective positive constants; one could proceed (cf equations (4) and (13)) by looking for $\tilde{v}_S \in (C^\infty(\overline{\Omega}))^3$ such that $\nabla \cdot \tilde{v}_S = S$ in Ω and $\tilde{v}_S = \tilde{v}_0$ on Γ, setting (cf equation (14)) $v = (\tilde{v}_S + \tilde{u})/\rho$, and considering the field $\tilde{u}$ as new unknown. All extra terms generated by the space dependence of these quantities do not seem to alter the mathematical structure of the problem; in fact, one could introduce a form which generalizes the form (28) and (still being bilinear, bounded and coercive) plays the same role; and these extra terms would not prevent the nonlinear operator appearing in equation (41) from remaining completely continuous. Of course, the condition for existence of at least one weak solution would change if plasma density, resistivity and viscosity coefficients are allowed to have a space dependence. As this question seems significant, we are currently working on it and shall present the results in a forthcoming research note.

5. Uniqueness of weak solutions

We conclude this study by dealing with the uniqueness of the solution. Suppose that condition (52) is satisfied, and that $\Phi = (u, B)$ and $\Phi' = (u', B')$ are two solutions; let us define $\Lambda \equiv \Phi - \Phi'$ ($\in V$). Thus, equation (37) is satisfied (for all $\Psi \in V$) by both Φ and Φ'; therefore, choosing $\Psi = \Lambda$ and remembering that U is linear, the following equality holds:

$$((U\Lambda, \Lambda)) = \mathcal{B}(\Phi', \Lambda) - \mathcal{B}(\Phi, \Lambda). \tag{54}$$

A proper calculation of the right-hand side of equality (54) is not straightforward at all; the result of this non-trivial step is the following (see appendix B):

$$
\begin{aligned}
B(\Phi', \Lambda) - B(\Phi, \Lambda) \\
&= \rho \int_\Omega \mathrm{d}^3x \, (\mathbf{\Lambda}_u \times \boldsymbol{v}_S) \cdot (\nabla \times \mathbf{\Lambda}_u) + \rho b(\mathbf{\Lambda}_u, \mathbf{\Lambda}_u, \boldsymbol{u}) \\
&\quad + \int_\Omega \mathrm{d}^3x \, (\boldsymbol{v}_S \times \mathbf{\Lambda}_B) \cdot (\nabla \times \mathbf{\Lambda}_B) + \int_\Omega \mathrm{d}^3x \, (\mathbf{\Lambda}_B \times \mathbf{\Lambda}_u) \cdot (\nabla \times \boldsymbol{B}') \\
&\quad + b(\mathbf{\Lambda}_B, \boldsymbol{u}', \mathbf{\Lambda}_B)
\end{aligned} \tag{55}
$$

where we have defined $\mathbf{\Lambda}_u \equiv \boldsymbol{u} - \boldsymbol{u}'$ and $\mathbf{\Lambda}_B \equiv \boldsymbol{B} - \boldsymbol{B}'$. Now, we use equations (45) and (54)–(55), and estimate the right-hand side of equality (55); we obtain

$$
\begin{aligned}
\|\Lambda\|^2 \leqslant \|\Lambda\|^2 \Bigg[& \|\boldsymbol{v}_S\|_{L^4(\Omega)} \left(\frac{3\sqrt{3}\rho M_1}{\mu_\star} + \frac{M_2}{\eta} \right) + \frac{\rho M_1^2}{\mu_\star^{3/2}} \|\Phi\| \\
& + \left(\frac{M_1 M_2}{\sqrt{\mu_\star}\eta} + \frac{M_2^2}{\sqrt{\mu_\star}\eta} \right) \|\Phi'\| \Bigg].
\end{aligned} \tag{56}
$$

For both Φ and Φ' estimate (53) holds; using it in the estimate (56), we obtain $\|\Lambda\|^2 \leqslant \|\Lambda\|^2 \chi$, with χ depending neither on Φ nor on Φ'. Thus, if $\chi < 1$, then $\|\Lambda\| = 0$, i.e. $\Lambda = 0$, i.e. $\Phi = \Phi'$: there exists only one solution. This condition is explicitly

$$
\begin{aligned}
& \|\boldsymbol{v}_S\|_{L^4(\Omega)} \left(\frac{3\sqrt{3}\rho M_1}{\mu_\star} + \frac{M_2}{\eta} \right) + \left(\frac{\rho M_1^2}{\mu_\star^{3/2}} + \frac{M_1 M_2}{\sqrt{\mu_\star}\eta} + \frac{M_2^2}{\sqrt{\mu_\star}\eta} \right) \\
& \qquad \times \frac{(1/\sqrt{\eta})\|\boldsymbol{v}_S\|_{L^4(\Omega)} \|\boldsymbol{B}_0\|_{L^4(\Omega)} + \|F_S\|}{1 - \|\boldsymbol{v}_S\|_{L^4(\Omega)} \left((3\sqrt{3}\rho M_1/\mu_\star) + (M_2/\eta) \right)} < 1.
\end{aligned} \tag{57}
$$

The requirement for uniqueness expressed by this formula is of the same kind as that for existence. It is important to remark, however, that condition (57) is more stringent than condition (52).

6. Conclusions

Based on the assumption that some difficulties in the ideal MHD model for toroidal equilibria may be surmounted by taking dissipative processes into account, we have analysed a general dissipative MHD model in which the nonlinearities are accounted for in a self-consistent way. The dissipative processes that we have considered are resistivity and viscosity as described by the Braginskii operator, concerning which we have shown that it has the expected (but, up to now, not proved) property of dissipating energy for any flow velocity field which does not vanish almost everywhere. Having established a problem for weak solutions, we have rigorously proved an existence and uniqueness theorem, and obtained an estimate of the solution(s). There exists at least one weak solution provided the dissipative processes are sufficiently strong, or the plasma source sustaining the pressure gradient is sufficiently small; uniqueness holds under a condition of the same kind, but more stringent.

Several questions seem to deserve further consideration and analysis. Although the existence and uniqueness conditions that we have obtained may turn out, because of the techniques which have had to be adopted to derive them, to be far too stringent, a numerical evaluation of them with parameters of interest for controlled fusion research would yield valuable insight. Two generalizations of the model analysed here would be significant:

(i) the account of more general boundary conditions than those relative to a perfectly conducting wall; and

(ii) to relinquish the assumption of uniform density which would become another unknown.

We point out that the latter generalization is not at all straightforward; in fact, one should clearly generalize existence and uniqueness theorems which hold for viscous compressible flows, the mathematical theory of which is of very great complexity and still rather incomplete. Finally, a (theoretical and computational) thorough analysis of bifurcation phenomena for this model would prove very significant and, we believe, even relevant to the interpretation of some aspects of the experimental results obtained in research on controlled thermonuclear fusion.

Appendix A. The hydrodynamic limit

The Navier–Stokes problem can manifestly be studied as a particular case of problem (15)–(19). One must consider:

(i) equation (15) in which: (a) $\boldsymbol{v}_S$, $\boldsymbol{B}_0$, $\boldsymbol{B}$ are set equal to zero; (b) $\hat{V}$ is replaced by $\mu_\star\Delta$; (c) the field $\boldsymbol{f}_S$ is replaced by a given, sufficiently regular, force field $\boldsymbol{f}$;

(ii) the former condition of equation (17); and

(iii) equation (18).

The different viscosity operator has no significant consequence for the analysis. In fact, assuming that $\boldsymbol{u}$ and $\boldsymbol{w}$ are smooth (cf equation (28) and what follows it), we have that $(-\mu_\star\Delta\boldsymbol{u}, \boldsymbol{w}) = \mu_\star(\partial_i\boldsymbol{u}, \partial_i\boldsymbol{w})$; thus, instead of definition (28), we ought to set $\mathcal{E}_{\mathrm{NS}}(\boldsymbol{a}, \boldsymbol{b}) \equiv \mu_\star((\boldsymbol{a}, \boldsymbol{b}))_1$. Since $\mathcal{E}_{\mathrm{NS}}$ is obviously bilinear and bounded, and satisfies equation (44), this assertion is manifestly true.

Therefore, equation (52) tells us immediately that for the Navier–Stokes problem at least one weak solution always exists. Note that it is the presence of the source which seems to prevent problem (15)–(19) from being solvable for small viscosity or resistivity (see Ladyzhenskaya (1963) on page xi).

As regards uniqueness, condition (57) clearly becomes $\rho M_1^2\|F\|/\mu_\star^{3/2} < 1$. Writing $F = (\boldsymbol{F}_u, \boldsymbol{0})$ we have that $\|F\| = \sqrt{\mu_\star}\|\boldsymbol{F}_u\|_1$; moreover, note that $(\boldsymbol{f}, \boldsymbol{w}) = ((F, \Psi)) = \mu_\star((\boldsymbol{F}_u, \boldsymbol{w}))_1 = ((\tilde{\boldsymbol{f}}, \boldsymbol{w}))_1$ where $\tilde{\boldsymbol{f}} \equiv \mu_\star\boldsymbol{F}_u$. Thus, the condition for uniqueness becomes $\rho M_1^2\|\tilde{\boldsymbol{f}}\|_1/\mu_\star^2 < 1$.

These important existence and uniqueness results for the Navier–Stokes problem are well known (Ladyzhenskaya 1963).

Appendix B. Elucidation of some non-trivial calculations

The derivation of equalities (49) and (55) is not straightforward. The main properties which must be used are the following:

(i) The form b is trilinear.

(ii) For all $\xi \in V_i$ $(i = 1, 2)$, and for all $\xi', \xi'' \in \boldsymbol{H}^1(\Omega)$ we have $b(\xi, \xi', \xi'') = -b(\xi, \xi'', \xi')$ and, in particular, $b(\xi, \xi', \xi') = 0$.

(iii) For all $\xi, \xi', \xi'' \in \boldsymbol{H}^1(\Omega)$ we have $b(\xi, \xi', \xi'') - b(\xi'', \xi', \xi) = \int_\Omega \mathrm{d}^3x\,(\xi \times \xi'') \cdot (\nabla \times \xi')$.

(iv) $\nabla \times \boldsymbol{B}_0 = \boldsymbol{0}$ in Ω.

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